

The University of Chicago

THE ASYMPTOTIC WARING PROBLEM
FOR HOMOGENEOUS POLYNOMIAL
SUMMANDS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1937

By
HAROLD CHATLAND

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BY H. CHATLAND

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1. Introduction. In this paper, by using homogeneous polynomials as summands, the number required to represent all sufficiently large integers is reduced to approximately two-thirds of that for n^{th} powers. The asymptotic Waring Problem for sums of n^{th} powers given by I. Vinogradow¹ was generalized and the result improved by L. E. Dickson,² who considered sums of the type

$$(1) \quad Ax^n + \sum_{j=1}^{s-3} A_j x_j^n + \sum_{i=1}^k a_i (w_i^n + W_i^n) + \sum_{i=1}^k a_{k+i} z_i^n y^n,$$

where the coefficients are given positive integers.

If $s \geq 4n$, the congruence

$$(2) \quad Ax^n + \sum_{j=1}^{s-3} A_j x_j^n + a_k (w^n + W^n) \equiv G \pmod{p^\gamma}$$

is known to have integral solutions such that not every term on the left is divisible by p , for every integer G and prime p and γ defined to be $\theta + 1$ if $p > 2$, $\theta + 2$ if $p = 2$, p^θ denoting the highest power of the prime p which divides n .

In this paper we treat the problem for sums of values of a homogeneous polynomial

$$(3) \quad Ax^n + \sum_{j=1}^{s-3} A_j x_j^n + \sum_{i=1}^{k+1} \{ \phi(x_i, y_i) + \phi(X_i, Y_i) \} + \sum_{i=1}^{k+1} y^n \phi(z_i, w_i),$$

where the coefficients are given positive integers and

$$(4) \quad \phi(x, y) = \sum_{r=0}^n b_r x^{n-r} y^r, \\ b_0 = b_1 = b_2 = 1, \quad 0 \leq b_i \leq (n+1)^{i-2} (n-1)^{i-3} \quad (i \geq 3).$$

The congruence corresponding to (2) is

$$(5) \quad Ax^n + \sum_{j=1}^{s-3} A_j x_j^n + (\phi(x_{k+1}, y_{k+1}) + \phi(X_{k+1}, Y_{k+1})) \equiv G \pmod{p^\gamma},$$

¹ I. Vinogradow, *On Waring's problem*, Annals of Mathematics, vol. 36 (1935), pp. 395-405.

² L. E. Dickson, *On Waring's problem and its generalization*, Annals of Mathematics, vol. 37 (1936), pp. 293-316.

which reduces to (2) when y_{k+1} , Y_{k+1} are zero. Hence (5) has solutions of the type described for (2) when $s \geq 4n$. Apart from constants, the argument is the same and the result holds when the coefficients a_i are put into the third and fourth sums of (3). We obtain the

THEOREM. *Every sufficiently large integer is a sum of $4n - 2 + 3k_0$ values of a homogeneous polynomial of the type described, where k_0 is the least integer k satisfying*

$$k > \frac{\log r}{-\log(1 - 3\nu/2)},$$

with $\nu = 1/n$, $r = n^2(6n - 7 + \nu)/(n - d)$, $d = 1 + 2n^2z$ and z is about $\nu^3/24$, $n \geq 4$.

The notations given by Vinogradov and Dickson are used in this paper. After several new steps in the theory we are able to use a part of Dickson's paper, with his notations having the new meaning given them for this problem.

Section 4 has no analogue in the problem for n^{th} powers and notations used there pertain only to that part.

2. Notations and definitions. Write

$$(6) \quad \nu = 1/n, \quad \sigma = (n - 1)(1 - 3\nu/2)^k.$$

Let c_i , h , and f denote positive numbers depending only upon n . Given a large integer N_0 , we take

$$(7) \quad P = [hN_0^\nu], \quad R = [P^{1-f}], \quad m = \max(3^{n-1}A, A_1, \dots, A_{s-3}),$$

$$(8) \quad P_1 = [(n/2^{n+1})^\nu P^{1-\nu}], \quad R_1 = \left[\left(\frac{n}{2^{n+1} - 1} \right)^\nu R^{1-\nu} \right], \quad \tau = 2mnP^{n-1/2},$$

$$(9) \quad Y = [C^\nu P^{(1-\nu)f}], \quad C = (3/2)^{n-1}.$$

Then

$$(10) \quad \tau \leq P^n \text{ if } P \geq (2mn)^2, \quad Y^n \leq \frac{1}{2}P^{\frac{1}{2}} \text{ if } P^{\frac{1}{2}-(n-1)f} \geq 2C.$$

The latter will be true when N_0 (and hence P) is sufficiently large and

$$(n - 1)f < \frac{1}{2}.$$

When z is real, $[z]$ denotes the distance from z to the nearest integer. When x is complicated, we denote e^x by $\text{Exp } x$. The largest integer $\leq x$ is denoted by $[x]$.

3. Five lemmas.³

LEMMA A. *If a , q , t are positive integers, $(a, q) = 1$, then*

$$\left| \sum_{r=0}^{q-1} \text{Exp } 2\pi i \frac{a}{q} tr^n \right| < cq^{1-\nu}, \quad c(n, t) > 0.$$

³ L. E. Dickson, *ibid.*

LEMMA B. When $s \geq 4n$, $t_j = 1$, $n \geq 4$, there exist positive constants $b = b(N, n, s)$, $c_5 = c_5(n, s)$, $c_2 = c_2(n, s)$, such that

$$(11) \quad \left| \sum_{q=1}^u A(q, N) - b \right| \leq c_5 u^{2-s\nu}, \quad b > c_2,$$

where for positive integers q, s, t_j ,

$$A(q, N) = \sum \text{Exp} \left(-2\pi i \frac{a}{q} N \right) \prod_{j=1}^s \frac{1}{q} \sum_{r=0}^{q-1} \text{Exp} 2\pi i \frac{a}{q} t_j r^n.$$

LEMMA B'. For every prime p and every integer G , let

$$\sum_{j=1}^s t_j r_j^n \equiv G \pmod{p^\gamma}$$

have a special solution. Let $s > 2n$, $s \geq 5$. Let t be the maximum of $t_1, \dots, t_s$. Then there exist positive constants $b = b(N, n, s, t)$, $c_i = c_i(n, s, t)$, satisfying (11).

LEMMA C. Let $0 \leq f'(y) \leq \frac{1}{2}$, $f''(y) \geq 0$ in the interval $g \leq y \leq h$, where g and h need not be integers. Then

$$\left| \sum_{g < y \leq h} \text{Exp} 2\pi i f(y) - \int_g^h \text{Exp} 2\pi i f(y) dy \right| < 5.$$

LEMMA D. Let λ be real, but not integral. Let G and H be integers, $G < H$. Then

$$\left| \sum_{x=G+1}^H \text{Exp} 2\pi i \lambda x \right| < 1/2[\lambda].$$

4. **Some inequalities.** We shall determine a rectangle in the xy -plane which is such that when the coördinates of certain points in it are substituted in the polynomial

$$(12) \quad \phi(x, y) = \sum_{r=0}^n b_r x^{n-r} y^r,$$

with $b_0 = b_1 = b_2 = 1$, $0 \leq b_i \leq (n+1)^{i-2}(n-1)^{i-3}$ ($3 \leq i \leq n$), the values of the polynomial will be distinct and the minimum difference between any two of them will be greater than a certain fixed quantity depending upon n and the least value P_1 of x in the rectangle considered. The following inequalities are necessary.

$$(I) \quad \phi(P_1, A(n-1)) > \phi(P_1 + n - 1, (A-n)(n-1)) \\ (n \leq A \leq P_1/(n+1)(n-1)^2),$$

in which A is a multiple of n . Compare i^{th} terms of the polynomials (I). We find that

$$(13) \quad b_i(n-1)^i A^i P_1^{n-i} > b_i(n-1)^i (A-n)^i (P_1 + n - 1)^{n-i}.$$

This written as a difference is

$$(14) \quad b_i(n-1)^i \{A^i P_1^{n-i} - (A-n)^i (P_1+n-1)^{n-i}\} > 0,$$

which is true if $\left(\frac{A}{A-n}\right)^i P_1^{n-i} > (P_1+n-1)^{n-i}$. Now $A/(A-n)$ is minimum for A maximum and hence for $A = P_1/(n+1)(n-1)^2$. Then

$$A/(A-n) = 1 + n/(A-n) > 1 + n/A = 1 + n(n+1)(n-1)^2/P_1.$$

Thus (14) holds if

$$(1+n(n+1)(n-1)^2/P_1)^i P_1^{n-i} > P_1^{n-i} + (n-i)(n-1)P_1^{n-i-1} + (\text{terms in lower powers of } P_1).$$

This is true since on expanding the left side and dropping P_1^{n-i} we find that the second term on the left is greater than the remaining part on the right. That is

$$in(n+1)(n-1)^2 P_1^{n-i-1} > (n-i)(n-1)P_1^{n-i-1} + \text{the remaining terms.}$$

Hence in (I) set $b_i = 0$ ($i \geq 3$). Subtracting the right side from the left we find the difference to be greater than

$$(A(n+1) - n(n+1)/2) P_1^{n-2} - (\text{terms in lower powers of } P_1).$$

Hence for P_1 large the difference between the right and left sides of (I) exceeds

$$(15) \quad A\left(\frac{n-1}{2}\right) P_1^{n-2}$$

for $n \leq A \leq P_1/(n+1)(n-1)^2$. When $A = Q$, where Q is the greatest multiple of n in $[P_1^{\frac{1}{3}}/(n-1)(n(n+1)/2)^{\frac{1}{3}}] + n$, the difference (15) is $> \frac{P_1^{n-3/2}}{2(n(n+1)/2)^{\frac{1}{3}}}$. When $A = n$, (15) is $\frac{n(n-1)}{2} P_1^{n-2}$.

The next inequality necessary is

$$(II) \quad \phi(P_1, An(n-1)) < \phi(P_1 + A(n-1) + 1, 0) \quad (0 \leq A \leq L_1 - 1),$$

where L_1 denotes $[n^{\frac{1}{3}} P_1^{\frac{1}{3}}/(n-1)(n(n+1)/2)^{\frac{1}{3}}]$. The left side of (II) does not exceed

$$(16) \quad P_1^n + An(n-1)P_1^{n-1} + A^2 n^2 (n-1)^2 P_1^{n-2} + (n+1)(n-1)^3 A^3 n^3 P_1^{n-3} + \lambda,$$

where λ denotes a sum of terms in lower powers of P_1 with b_i ($i \geq 4$) having their maximum values given below (12). Replace the left side of (II) by (16) and subtract from the right side of (II). The difference exceeds

$$(17) \quad nP_1^{n-1} - P_1^{n-2}(n-1)^2(A^2 n(n+1)/2 - An) - A^3 n^3 (n+1)(n-1)^3 P_1^{n-3}.$$

This follows since the fourth term on the right side, which contains a higher power of P_1 than does λ , has been discarded along with the terms following it.

The difference (17) is minimum when A takes its maximum value $L_1 - 1$. When $A = L_1 - 1$, (17) is greater than $(n - 1)n\left(\frac{n + 1}{2}\right)^{\frac{1}{2}} P_1^{n-3/2}$.

Let y take the values $0, n(n - 1), 2n(n - 1), 3n(n - 1), \dots$. Let $\Phi(P_1, i) \equiv \phi(P_1, in(n - 1))$. Let A in (I) be denoted by A_1 . If y takes the value $jn(n - 1)$ then A_1 must be a multiple jn of n . Thus (I) becomes

$$(I') \quad \Phi(P_1, j) > \Phi(P_1 + (n - 1), j - 1) \quad \left(j = 1, 2, \dots, \frac{P_1}{n(n + 1)(n - 1)^2} \right).$$

Since in (II) the y -coördinates are multiples of n , we may write (II) as

$$(II') \quad \Phi(P_1, A) < \Phi(P_1 + A(n - 1) + 1, 0) \quad (1 \leq A \leq L_1 - 1).$$

By (I')

$$\begin{aligned} \Phi(P_1 + A(n - 1) + 1, 0) &< \Phi(P_1 + (A - 1)(n - 1) + 1, 1) \\ &< \dots < \Phi(P_1 + (A - i)(n - 1) + 1, i). \end{aligned}$$

By (II') and this

$$\Phi(P_1, A) < \Phi(P_1 + (A - i)(n - 1) + 1, i) \quad (i = 0, \dots, A).$$

In the proof of (II) we found that the difference between the right and left sides exceeds $(n - 1)n\left(\frac{n + 1}{2}\right)^{\frac{1}{2}} P_1^{n-3/2}$. Hence the difference between the two sides of the last inequality is greater than $(n - 1)n\left(\frac{n + 1}{2}\right)^{\frac{1}{2}} P_1^{n-3/2}$ for all values of i . By (I')

$$\Phi(P_1, A) > \Phi(P_1 + (n - 1), A - 1) > \dots > \Phi(P_1 + k(n - 1), A - k),$$

which becomes, when $k = A - i$,

$$\Phi(P_1, A) > \Phi(P_1 + (A - i)(n - 1), i) \quad (i = Q/n, Q/n + 1, \dots, A).$$

We restrict i to begin at Q/n (where Q as we have stated above is the greatest multiple of n in $[P_1^{\frac{1}{2}}/(n - 1)(n(n + 1)/2)^{\frac{1}{2}}] + n$) in order that the difference between the two sides of the last inequality exceed

$$\frac{(n - 1)}{2} \left(\frac{P_1^{n-3/2}}{(n(n + 1)/2)^{\frac{1}{2}}(n - 1)} \right)$$

for the values of i given. Thus

$$(III) \quad \begin{aligned} \Phi(P_1 + (A - i)(n - 1), i) &< \Phi(P_1, A) \\ &< \Phi(P_1 + (A - i)(n - 1) + 1, i) \quad (i = Q/n, \dots, A - 1), \end{aligned}$$

and in which $A \leq L_1 - 1$. Hence the values of $\phi(x, y)$ at the points

$$(P_1 + k, in(n - 1)) \quad (0 \leq k \leq P_1, Q/n \leq i \leq L_1 - 1)$$

will all be distinct. For, by (III) the value of $\phi(x, y)$ at any point in this rectangle lies between the values at two successive points in the same horizontal line determined by i , for every i . Also, in any horizontal line $\phi(x, y)$ is strictly increasing with x .

The number of points (and hence of distinct values of ϕ) in this rectangle is $(L_1 - Q/n)P_1 > (n^{\frac{1}{2}} - 1)P_1^{3/2}/(n - 1)\left(\frac{n(n+1)}{2}\right)^{\frac{1}{2}}$. The difference between any two values of $\phi(x, y)$ in the rectangle exceeds $P_1^{n-3/2}/(2n(n+1))^{\frac{1}{2}} = c_1 P_1^{n-3/2}$.

5. The numbers ξ and u . In the last part we determined for P_1 a set of points (x, y) at which $\phi(x, y)$ takes on distinct values. These points lie in the rectangle

$$P_1 \leq x \leq 2P_1, \quad c_4 P_1^{\frac{1}{2}} \leq y < c_0 P_1^{\frac{1}{2}},$$

where $c_4 P_1^{\frac{1}{2}} = [(1/n)[P_1^{\frac{1}{2}}/(n(n+1)/2)^{\frac{1}{2}}(n-1)] + 1](n-1)$,

$$c_0 P_1^{\frac{1}{2}} = [n^{\frac{1}{2}} P_1^{\frac{1}{2}}/(n(n+1)/2)^{\frac{1}{2}}(n-1)]n(n-1).$$

Let $P_i = [\gamma^{i-1} P_1^{(1-3\nu/2)^{i-1}}]$ ($i = 1, \dots, k$), where $\gamma = 1$ if $(c_1/kc_3)^r 2^{3\nu/2-1} > 1$, otherwise $\gamma = (c_1/kc_3)^r 2^{3\nu/2-2}$ ($c_3 = 2^n + 1$), whence γ is independent of i and P_1 .

For each P_i , as for P_1 , a rectangle Δ_i can be determined containing a set of points (x, y) at which $\phi(x, y)$ takes on distinct values. As before, for each such set the difference between any two distinct values of $\phi(x, y)$ exceeds $c_1 P_i^{n-3/2}$. Also

$$\phi(x, y) \leq \phi(2P_i, c_0 P_i^{\frac{1}{2}}) \leq c_3 P_i^n.$$

Let (x_i, y_i) and (z_i, w_i) be in Δ_i and distinct. We shall prove

$$(18) \quad \phi(x_i, y_i) - \phi(z_i, w_i) > c_1 P_i^{n-3/2} > k\phi(2P_{i+1}, c_0 P_{i+1}^{\frac{1}{2}}) \geq \sum_{j=i+1}^k \phi(x_j, y_j),$$

where (x_j, y_j) lie in Δ_j ($j > i$).

The first inequality in (18) follows from the minimum difference. The rectangles are such that the maximum $\phi(x, y)$ with (x, y) in Δ_{i+1} is greater than the maximum of $\phi(x, y)$ with (x, y) chosen from any succeeding set Δ . Moreover the Δ_i ($i = 1, \dots, k$) are such that the second inequality holds. Note that the maximum value of $\phi(x_j, y_j)$ ($j \geq i+1$) is attained at $(2P_j, c_0 P_j^{\frac{1}{2}})$ and there

$$\phi(x_l, y_l) < c_3 P_l^n < 2^n P_{i+1}^n < \phi(x_{i+1}, y_{i+1}) < c_3 P_{i+1}^n \quad (l > i+1).$$

Then

$$kc_3 P_{i+1}^n > \sum_{j=i+1}^k \phi(x_j, y_j),$$

and

$$k\phi(2P_{i+1}, c_0 P_{i+1}^{\frac{1}{2}}) < kc_3 P_{i+1}^n.$$

Hence the second inequality in (18) holds if

$$P_{i+1} < (c_1/kc_3)^v P_i^{1-3v/2}.$$

This will follow if

$$P_{i+1} \leq \gamma^i P_1^{(1-3v/2)i},$$

and

$$P_i > \frac{1}{2} \gamma^{i-1} P_1^{(1-3v/2)(i-1)}$$

for P_1 large. Now

$$P_{i+1} < (c_1/kc_3)^v P_i^{1-3v/2}$$

if

$$\gamma^i P_1^{(1-3v/2)i} < (\frac{1}{2} \gamma^{i-1})^{1-3v/2} (c_1/kc_3)^v P_1^{(1-3v/2)i}.$$

That is, if

$$\gamma^{i-(i-1)(1-3v/2)} < (c_1/kc_3)^v 2^{3v/2-1} \quad (i = 1, \dots, k).$$

If the inequality is true when $i = 1$, it will hold for $i > 1$ provided $\gamma \leq 1$. When $i = 1$, it is satisfied as follows. If $(c_1/kc_3)^v 2^{3v/2-1} \leq 1$, $\gamma < (c_1/kc_3)^v 2^{3v/2-1}$, i.e. $\gamma = (c_1/kc_3)^v 2^{3v/2-2}$. If $(c_1/kc_3)^v 2^{3v/2-1} > 1$, $\gamma = 1$.

LEMMA.⁴ If we have an integer N such that

$$N = \phi(x_1, y_1) + \phi(x_2, y_2) + \dots + \phi(x_i, y_i) + \dots + \phi(x_k, y_k),$$

with x_i, y_i in Δ_i then this representation of N is unique.

For, if also

$$N = \phi(x_1, y_1) + \dots + \phi(x_{i-1}, y_{i-1}) + \phi(x'_i, y'_i) + \dots + \phi(x'_k, y'_k),$$

with x'_i, y'_i in Δ_i but distinct from x_i, y_i , then by (18)

$$\phi(x_i, y_i) > \phi(x'_i, y'_i) + \dots + \phi(x'_k, y'_k).$$

Hence we reach a contradiction.

To form the numbers ξ , take all possible sums of k values of $\phi(x, y)$, one value chosen from each rectangle of sets (x, y) . These are distinct by the last lemma. The number of ξ 's will be the product of the number of values arising from each rectangle. Hence if X is the number of ξ 's,

$$X > \chi (P_1^{3/2})^{\sum_{i=1}^k (1-3v/2)i} = \chi P_1^{\frac{3}{2} \left(\frac{1-(1-3v/2)k}{3v/2} \right)} = \chi P_1^{n-n(1-3v/2)k},$$

where χ is a product of constants. Hence by (8)

$$X > \chi (n/2^{n+2})^{1-(1-3v/2)k} P^{(n-1)(1-(1-3v/2)k)},$$

which by (6) is

$$(19) \quad X > \chi (n/2^{n+2})^{1-(1-3v/2)k} P^{n-1-\sigma}.$$

⁴ Proved in a special case by H. Davenport and H. Heilbronn, *On Waring's theorem for fourth powers*, Proceedings of the London Mathematical Society, vol. 41 (1936), p. 143.

The integers u defined by

$$(20) \quad u = \xi + v^n \quad (v = P, P + 1, \dots, 2P - 1)$$

will be proved to be distinct. By (8)

$$(21) \quad nP^{n-1} \geq P_1^n(2^{n+1}).$$

Also

$$(P + j)^n > P^n + nP^{n-1} \text{ for } j \geq 1.$$

Now ξ maximum does not exceed $\phi(2P_1, c_0P_1^{\frac{1}{2}} + k2^{n+1}P_1^{n-3/2})$, and hence is less than $2^{n+1}P_1^n$. Hence since

$$(P + j)^n > (P^n + nP^{n-1}) \geq P^n + 2^{n+1}P_1^n,$$

and

$$(P + 1)^n - P^n < (P + j)^n - (P + j - 1)^n \quad j > 1,$$

the u 's are all distinct. The maximum u is less than

$$(2P - 1)^n + 2^{n+1}P_1^n < (2P - 1)^n + nP^{n-1} < (2P)^n.$$

The minimum u is greater than P^n . Hence

$$(22) \quad P^n < u < (2P)^n.$$

Just as we defined the numbers ξ and u by use of P, P_1 , we now define ξ_1 and u_1 by use of R, R_1 . Under these replacements let χ become χ_1 . Thus if X_1 is the number of ξ_1 's, $X_1 > \chi_1(n/2^{n+2})^{1-(1-3\nu/2)^k}R^{n-1-\sigma}$. Hence

$$(23) \quad u_1 = \xi_1 + v_1^n \quad (v_1 = R, \dots, 2R - 1),$$

$$(24) \quad X_1 > \chi_1(n/2^{n+2})^{1-(1-3\nu/2)^k}R^{n-1-\sigma},$$

$$(25) \quad R^n < u_1 < (2R)^n.$$

6. The fundamental integral. Take $N = N_0$ and consider

$$(88) \quad I_{N_0} = \sum_{y=1}^Y I_{yN_0} = \int_{-r^{-1}}^{1-r^{-1}} T_1 \dots T_{s-3} TV^2 S^2 \sum_y V_y S_y e^{-2\pi i \alpha N_0} d\alpha.$$

As in Dickson we obtain

$$(89) \quad I_{N_0} = H_1 + H_2, \quad H_1 = \sum_y H_{y1} > c_9 Y X^2 X_1 R P^{s-n},$$

$$(96) \quad |H_2| < D P^{s-2} P X (X_1 R Y P^{s+n-1})^{\frac{1}{2}}, \quad D = 3(8mn)^{\frac{1}{2}},$$

with his k replaced by $k + 1$. The constants arise in the same way as in his paper with values appropriate to this paper. The formulas are numbered as in his paper. By (7), (9), (19), (24),

$$(97) \quad R > \sqrt{\frac{1}{2}} P^{1-f}, \quad Y > \sqrt{\frac{1}{2}} C^v P^{(1-v)f},$$

$$X \geq (n/2^{n+2})^{1-(1-3\nu/2)^k} \chi P^{n-1-\sigma}$$

while X_1 exceeds the last products with P replaced by R and χ_- replaced by χ_1 . Note that

$$(98) \quad \sigma = (n-1)(1-3\nu/2)^k, \quad \mu = 1 - (1-3\nu/2)^k.$$

Multiply and divide (96) by $X(X_1RY)^{\frac{1}{2}}$ and apply (97). Thus

$$(99) \quad |H_2| < C_1 P^J X^2 X_1 R Y P^{s-n}, \quad C_1 = \frac{2^{(n+1-\sigma)/4} D}{\chi(n/2^{n+2})^\mu \{C^\nu \chi_1(n/2^{n+2})^\mu\}^{\frac{1}{4}}},$$

$$(100) \quad 2J = \sigma(3-f) - g, \quad g = \frac{1}{2} - z - nf + (1-\nu)f,$$

The Waring theorem is true for every integer $\geq N_0$ if the integral $I_{N_0} > 0$. By the above and (89), this is true if

$$(101) \quad P^{-J} > C_1/c_0.$$

For P large, this holds if $J < 0$. This is true by (98₂) if

$$(102) \quad r(1-3\nu/2)^k < 1, \quad r = (n-1)(3-f)/g.$$

But r increases with f since

$$\frac{dr}{df} = \left(\frac{n-1}{g^2} \right) (3n-7/2+z+3\nu).$$

Since (102₁) is equivalent to

$$(103) \quad k > \frac{\log r}{-\log(1-3\nu/2)},$$

k will be minimum if r and hence f is. But $f \geq \nu/2$. Hence we employ the minimum f , viz.,

$$(104) \quad f = \nu/2.$$

The resulting value of r in (102) is

$$(105) \quad r = n^2(6n-7+\nu)/(n-d), \quad d = 1 + 2n^2z.$$

For a small z , say $\nu^3/24$, let k_0 be the least integer k satisfying (103). Then all sufficiently large integers are sums of $4n-2+3k_0$ values of a homogeneous polynomial of the type previously described.

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